OPTIMAL CONTROL APPLICATIONS AND METHODS
Optim. Control Appl. Meth. (2011)
Published online in Wiley Online Library (wileyonlinelibrary.com). DOI: 10.1002/oca.1023

Reliable control for interval time-varying delay systems subjected to actuator saturation and stochastic failure

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SUMMARY

This paper is devoted to the problem of reliable control for interval time-varying delay systems subjected to actuator saturation and stochastic failure. A new practical actuator fault model is proposed by assuming that the actuator fault obeys a certain probabilistic distribution. An optimization problem with LMI constraints is formulated to determine the largest contractively invariant ellipsoid. Delay distribution and fault distribution-dependent estimations of the domain of attraction are obtained by using the LMI techniques and an optimization method, such that the mean-square stability of the systems can be guaranteed for given H_∞ performance index γ. Two illustrative examples are exploited to show the effectiveness of the proposed design procedures. Copyright © 2011 John Wiley & Sons, Ltd.

Received 2 October 2010; Revised 19 May 2011; Accepted 14 September 2011

KEY WORDS: reliable control; time-varying delay; stochastic fault model; domain of attraction

1. INTRODUCTION

Actuator saturation is usually encountered by reasons of physical constraints. It may have a deleterious effect on both the stability and performance of controlled systems. Considerable attention has been devoted to the kind of liner system subject to saturating controller. Two main approaches have been developed in the existing literature: (i) let saturation do not occur, which is called positive invariance approach [1]; (ii) allow saturations to take effect while guaranteeing asymptotic stability [2, 3]. The main challenge of those two main approaches is to obtain a large enough domain of initial states which ensures asymptotic stability for the system despite the presence of saturations.

On the other hand, much effort has been devoted to the reliable control for time-delay systems. Firstly, the existence of the time delay may cause instability or bad performances in dynamic systems. Hence, the stability and stabilization problems for time delay systems have received some attenuation [4–6] and the references therein. Secondly, unexpected faults or failures may result in substantial damage[7–9]. A high degree of fault tolerance for the operational systems is an essential and integrated part of the overall control system design. It is noted that the reliable control designing methods in the open literatures are based on the assumption that control component failures are modeled as outages [10, 11]; that is, when a failure occurs, the actuators signal simply becomes zero; or modeled as partly outage [12, 13], that is, the control input cannot reach its full gain but work in an exact amplitude. However, it cannot represent actuator-failure exactly. The actuator may not be completely failure; that is, the scale factor $\xi_i = 0$ is the simplest special cases. In practical systems, because of actuators aging, zero shift, Electromagnetic Interference, nonlinear amplification

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in different frequency field, and so on, it will be more reasonable that the fault scale factor obeys a certain probabilistic distribution in an interval. To the best of our knowledge, it seems that there are no results on the problem of reliable control with such an actuator fault model which satisfies a certain probabilistic distribution. This motivates us to further investigate the problem of reliable control systems with stochastic actuators failures.

This paper investigates the reliable H_∞ control design for interval time-varying delay systems with consideration of both stochastic actuator failures and saturations. The failure scale factor of each actuator is governed by a random variable; a new stochastic fault model is proposed. Delay distribution and fault distribution-dependent approaches are adopted, and the corresponding existence criterion of the stabilizing controller is derived via LMI formulation. Furthermore, an optimization problem with LMI constraints is formulated to obtain the largest contractively invariant set. Two numerical examples are given to show the effectiveness of the proposed design procedures.

Notation: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ is the set of real $n \times m$ matrices, I is the identity matrix of appropriate dimensions, and $\|\cdot\|$ stands for the Euclidean vector norm or spectral norm as appropriate. The notation $X > 0$ (respectively, $X < 0$), for $X \in \mathbb{R}^{n\times n}$ means that the matrix X is a real symmetric positive definite (respectively, negative definite). $\mathscr{C}_{n,\tau} = \mathscr{C}([-\tau\ 0], \mathbb{R}^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau\ 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. When x is a stochastic variable, $\mathbb{E}\{x\}$ stands for the expectation of x. $\mathscr{L}$ denotes infinitesimal operator. The asterisk $*$ in a matrix is used to denote term that is induced by symmetry; matrices, if they are not explicitly stated, are assumed to have compatible dimensions.

2. SYSTEM DESCRIPTION

In this paper, we consider a class of interval time-varying delay systems with actuator saturation

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-\tau(t)) + B_1\sigma(u(t)) + B_2\omega(t) \\ y(t) = Cx(t) + D\omega(t) \\ x(s) = \phi(s), s \in [-\tau_2 - \tau_1], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ denote the state and control vectors, respectively. $\omega(t)$ is the disturbance. A, A_d, B_1, and B_2 are known constant matrices with appropriate dimensions, $\tau(t)$ is a time-varying delay which satisfies $\tau_1 \leqslant \tau(t) \leqslant \tau_2$, $\phi(s)$ is a continuous vector valued initial function. The function $\sigma(\cdot)$ is the standard saturation function defined as follows:

$$\sigma(u(t)) = [\sigma(u_1(t))\sigma(u_2(t))\cdots\sigma(u_m(t))]^T, \tag{2}$$

and

$$\sigma(u_i(t)) \triangleq \begin{cases} \bar{u}_i \text{ if } u_i(t) > \bar{u}_i \\ u_i(t) \text{ if } -\bar{u}_i < u_i(t) < \bar{u}_i \quad i = 1, 2, \cdots, m. \\ -\bar{u}_i \text{ if } u_i(t) < \bar{u}_i \end{cases} \tag{3}$$

Here, a more general actuator fault model for the system (1) is proposed

$$u^F(t) = \Xi\sigma(u(t)), \tag{4}$$

where $\Xi = diag\{\xi_1, \cdots, \xi_m\}$, and ξ_i $(i = 1, \cdots, m)$ are m unrelated random variables which denote the fault scale factor of each channel. The mathematical expectation and variance of ξ_i are μ_i and δ_i^2, respectively. For convenience, we define $\bar{\Xi} = diag\{\xi_1, \cdots, \xi_m\}$.

Remark 1

Equation (4) describes the actuator fault by a random matrix ξ_i which satisfies a certain probabilistic distribution in an interval. In particular, if the case $\xi_i = 0$, it stands for an entire missing of signals, and if $\xi_i = 1$, it indicates intactness. In fact, actuator signal drift usually occurs in practice situations, whereas completely failure and intactness are only two special cases.

DOI: 10.1002/oca

For convenience, we define $F_i = diag\{\underbrace{0,\cdots,0}_{i-1},1,\underbrace{0,\cdots,0}_{m-i}\}$. Then, the control input with actuator failure can be rewritten as

$$u^F(t) = \sum_{i=1}^{m} \xi_i F_i \sigma(Kx(t)). \tag{5}$$

Remark 2
In [14], defining $u^F(t) = (1-\rho)\sigma(u)$, where ρ $(0 < \rho < 1)$ is a scalar, which denotes a fault scale factor. However, each control input channel should be with different fault scale factor. In this paper, a rand diagonal matrix Ξ is introduced to denote every channel's fault.

Remark 3
In order to reduce the conservativeness by reasons of actuators fault, we can introduce some probability information, for example mathematical expectation and variance, and so on, in criterions by applying statistics method.

Then, the dynamics of (1) with actuator fault model (5) can be described by

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-\tau(t)) + B_1 \Xi \sigma(Kx(t)) + B_2 \omega(t) \\ z(t) = Cx(t) + D\omega(t) \\ x(s) = \phi(s), s \in [-\tau_2 - \tau_1]. \end{cases} \tag{6}$$

To estimate for the domain of attraction, the following subset are introduced.

$$\mathcal{L}(K) \triangleq \{x(t) \in \mathbb{R}^n : |k_i x(t)| \leqslant \bar{u}_i, i \in \mathcal{I}\}, \tag{7}$$

where k_i is the i-th row of the matrix K, and

$$\mathscr{E}(P,1) \triangleq \{x(t) \in \mathbb{R}^n : x(t)^T P x(t) \leqslant 1\}, \tag{8}$$

which is a contractively invariant ellipsoid.

The objective of this study is to develop a reliable controller for the closed-loop system with consideration for both stochastic actuator failure and actuator saturation. For this purpose, the following lemmas and definitions are introduced.

Definition 1 ([15]*)*
For an initial condition $x_0 = \phi \in \mathscr{C}_{n,\tau_2}$, suppose the state trajectory of system (6) $x(t, x_0)$ is the mean square asymptotically stable, the domain of attraction of the origin is then defined as

$$\mathscr{X} := \{x_0 \in \mathscr{C}_{n,\tau_2} : \lim_{t\to\infty} \mathbb{E}\{x(t,x_0)\}\} = 0\}. \tag{9}$$

Lemma 1 ([16]*)*
For any constant matrix $R \in \mathbb{R}^{n\times n}$, $R > 0$, scalar $\tau_m > 0$, and vector function $\dot{x} : [-\tau_m, 0] \to \mathbb{R}^n$ such that the following integration is well defined, it holds that

$$-\tau_m \int_{t-\tau_m}^{t} \dot{x}^T(t) R \dot{x}(t) \leqslant \begin{bmatrix} x(t) \\ x(t-\tau_m) \end{bmatrix}^T \begin{bmatrix} -R & R \\ * & -R \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau_m) \end{bmatrix}.$$

Lemma 2 ([6]*)*
Suppose M, N, and Ω are constant matrices of appropriate dimensions. Then,

$$(\tau(t) - \tau_m)M + (\tau_M - \tau(t))N + \Omega < 0, \tag{10}$$

is true for any $\tau(t) \in [\tau_m \quad \tau_M]$ if and only if

$$(\tau_M - \tau_m)M + \Omega < 0, \tag{11}$$

DOI: 10.1002/oca

$$(\tau_M - \tau_m)N + \Omega < 0. \tag{12}$$

Let $\mathscr{W}$ be the set of $m \times m$ diagonal matrices whose diagonal elements are 1 or 0. W_i $(i = 1, 2, \cdots, 2^m)$ is the element of $\mathscr{W}$, and define $W_i^- = I - W_i$, obviously, W_i^- is also an element of $\mathscr{W}$.

Lemma 3

Let $K \in \mathbb{R}^{m \times n}$ be given. For $x(t) \in \mathbb{R}^n$, if $x \in \mathcal{L}(H)$, then

$$\sigma(Kx(t)) \in co\{W_i Kx(t) + W_i^- Hx(t) : i \in \mathcal{I}\}, \tag{13}$$

where the notation $co\{\cdot\}$ denotes the convex hull of a set.

3. MAIN RESULT

In this section, the region of local mean-square stability, associated to the systems with actuator saturation and stochastic failures is considered firstly, and the design of the reliable control is then presented to tolerate the saturations and failures.

Theorem 1

For given scalars $\tau_1, \tau_2, \delta_i (i \in \mathcal{I})$ and matrix $\bar{\Xi}$, the closed-loop system (6) under all possible faults and within the set $\mathscr{E}(P, 1)$ is mean-square asymptotically stable, if there exist matrices $P > 0, Q_1 > 0, Q_2 > 0, R_1 > 0$ and $R_2 > 0$ such that the following hold

$$\mathscr{E}(P, 1) \subset \mathcal{L}(H), \tag{14}$$

$$\begin{bmatrix} \Psi_i^{11} + \Phi + \Phi^T & * & * & * \\ \mathscr{R}\Psi_i^{21} & -\mathscr{R} & * & * \\ \mathcal{R}\Psi_i^{31} & 0 & -\mathcal{R} & * \\ \Psi^{41}(r) & 0 & 0 & -R_2 \end{bmatrix} < 0 \quad (i = 1, 2, \cdots, 2^m, r = 1, 2), \tag{15}$$

where

$$\Psi_i^{11} = \begin{bmatrix} \Gamma_i & * & * & * & * \\ R_1 & -Q_1 - R_1 & * & * & * \\ A_d^T P & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \\ B_2^T P & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix},$$

$$\Gamma_i = (A + B_1\bar{\Xi}\mathcal{K}_i)^T P + P(A + B_1\bar{\Xi}\mathcal{K}_i) + Q_1 + Q_2 - R_1 + C^T C,$$

$$\Phi = \begin{bmatrix} 0 & M & N - M & -N & 0 \end{bmatrix},$$

$$\Psi_i^{21} = \begin{bmatrix} (A + B_1\bar{\Xi}\mathcal{K}_i) & 0 & A_d & 0 & B_2 \end{bmatrix},$$

$$\Psi_i^{31} = \begin{bmatrix} \delta_1 B_1 F_1 \mathcal{K}_i & 0 & 0 & 0 & 0 \\ \vdots & & & \vdots & \\ \delta_m B_1 F_m \mathcal{K}_i & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{R} = diag\{\underbrace{\mathscr{R} \cdots \mathscr{R}}_{m}\}, \mathscr{R} = \tau_1^2 R_1 + (\tau_2 - \tau_1)R_2,$$

$$\Psi^{41}(1) = \sqrt{\tau_2 - \tau_1} M^T, \Psi^{41}(2) = \sqrt{\tau_2 - \tau_1} N^T.$$

DOI: 10.1002/oca

Proof
Choose a Lyapunov function as

$$V(x_t) = x^T(t)Px(t) + \int_{t-\tau_1}^{t} x^T(s)Q_1x(s)ds + \int_{t-\tau_2}^{t} x^T(s)Q_2x(s)ds \\ + \tau_1 \int_{-\tau_1}^{0}\int_{t+s}^{t} \dot{x}^T(v)R_1\dot{x}(v)dvds + \int_{-\tau_2}^{-\tau_1}\int_{t+s}^{t} \dot{x}^T(v)R_2\dot{x}(v)dvds.$$

Using the infinitesimal operator for $V(x_t)$, it follows

$$\mathscr{L}V(x_t) = \mathbb{E}\left\{2x^T(t)P\dot{x}(t) + x^T(t)(Q_1+Q_2)x(t) - \sum_{i=1}^{2} x^T(t-\tau_i)R_ix(t-\tau_i) \right. \\ \left. + \dot{x}^T(t)\mathscr{R}\dot{x}(t) - \tau_1\int_{t-\tau_1}^{t} \dot{x}^T(s)Q_1\dot{x}(s)ds - \int_{t-\tau_2}^{t-\tau_1} \dot{x}^T(s)R_2\dot{x}(s)ds\right\}. \quad (16)$$

Employing the free-weighting matrix method [17], we have

$$2\zeta^T(t)M\left[x(t-\tau_1) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t-\tau_1} \dot{x}^T(s)ds\right] = 0, \quad (17)$$

$$2\zeta^T(t)N\left[x(t-\tau(t)) - x(t-\tau_2) - \int_{t-\tau_2}^{t-\tau(t)} \dot{x}^T(s)ds\right] = 0, \quad (18)$$

where $\zeta(t) = \begin{bmatrix} x^T(t) & x^T(t-\tau_1) & x^T(t-\tau(t)) & x^T(t-\tau_2)\end{bmatrix}^T$.
Note that

$$-2\zeta^T(t)M\int_{t-\tau(t)}^{t-\tau_1} \dot{x}(s)ds \leqslant (\tau(t)-\tau_1)\zeta^T(t)MR_2^{-1}M^T\zeta(t) + \int_{t-\tau(t)}^{t-\tau_1} \dot{x}^T(s)R_2\dot{x}(s)ds, \quad (19)$$

$$-2\zeta^T(t)N\int_{t-\tau_2}^{t-\tau(t)} \dot{x}(s)ds \leqslant (\tau_2-\tau(t))\zeta^T(t)NR_2^{-1}N^T\zeta(t) + \int_{t-\tau_2}^{t-\tau(t)} \dot{x}^T(s)R_2\dot{x}(s)ds. \quad (20)$$

According to (14) and Lemma 3, $\sigma(u(t))$ can be rewritten as

$$\sigma(Kx(t)) = \sum_{i=1}^{2^m} \theta_i\mathcal{K}_ix(t), (\sum_{i=1}^{2^m}\theta_i = 1, \theta_i > 0, i \in \mathcal{I}), \quad (21)$$

where $\mathcal{K}_i = (W_iK + W_i^-H)$.
By the definition of the matrix Ξ in (5), we can further rewrite the closed-loop system as

$$\dot{x}(t) = \sum_{i=1}^{2^m}\theta_i\left[(A + B_1\bar{\Xi}\mathcal{K}_ix(t) + B_1(\Xi-\bar{\Xi})\mathcal{K}_ix(t) + A_dx(t-\tau(t)) + B_2\omega(t)\right], \quad (22)$$

where $\bar{\Xi}$ is the expeditions of Ξ.

DOI: 10.1002/oca

Combining (16)–(22) and utilizing Lemma 1, we can obtain

$$\begin{aligned}\mathscr{L}V(x_t) \leqslant \mathbb{E}\Bigg\{ & \sum_{i=1}^{2^m} \theta_i \left\{2x^T(t)P(A+B_1\bar{\Xi}\mathcal{K}_i)x(t) + 2x^T(t)PA_d x(t-\tau(t)) + 2x^T(t)PB_2\omega(t)\right. \\ & + x^T(t)(Q_1+Q_2)x(t) - \sum_{i=1}^{2} x^T(t-\tau_i)Q_i x(t-\tau_i) \\ & + \dot{x}^T(t)\mathscr{R}\dot{x}^T(t) + \begin{bmatrix} x(t) \\ x(t-\tau_1)\end{bmatrix}^T \begin{bmatrix} -R_1 & * \\ R_1 & -R_1 \end{bmatrix}\begin{bmatrix} x(t) \\ x(t-\tau_1)\end{bmatrix} \\ & + 2\zeta^T(t)\left[Mx(t-\tau_1) + (N-M)x(t-\tau(t)) - Nx(t-\tau_2)\right] \\ & + \left.\zeta^T(t)\left[(\tau(t)-\tau_1)MR_2^{-1}M^T + (\tau_2-\tau(t))NR_2^{-1}N^T\right]\zeta(t)\right\}\Bigg\}. \end{aligned} \tag{23}$$

Recalling the definition of Ξ and $\bar{\Xi}$, we have

$$\begin{aligned}\mathbb{E}\left\{\sum_{i=1}^{2^m}\theta_i\left\{\dot{x}^T(t)\mathscr{R}\dot{x}(t)\right\}\right\} = \mathbb{E}\Bigg\{\sum_{i=1}^{2^m}\Bigg\{ & \eta^T(t)\Psi_i^{21^T}\mathscr{R}\Psi_i^{21}\eta(t) \\ & + \sum_{j=1}^{m}\delta_j^2 x^T(t)(B_1F_j\mathcal{K}_i)^T\mathscr{R}(B_1F_j\mathcal{K}_i)x(t)\Bigg\}\Bigg\}, \end{aligned} \tag{24}$$

where $\Psi_i^{21} = \left[(A+B_1\bar{\Xi}\mathcal{K}) \quad 0 \quad A_d \quad 0 \quad B_2\right]$, $\eta(t) = [\zeta^T(t)\omega^T(t)]^T$.

Substituting (24) into (23), we can obtain

$$\begin{aligned} & \mathscr{L}V(x_t) + \mathbb{E}\{z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t)\} \\ & \leqslant \mathbb{E}\Bigg\{\sum_{i=1}^{2^m}\eta^T(t)\left\{\Psi_i^{11} + \Phi + \Phi^T + \Psi_i^{21^T}\mathscr{R}\Psi_i^{21}\right. \\ & \qquad + \left.\left[(\tau(t)-\tau_1)MR_2^{-1}M^T + (\tau_2-\tau(t))NR_2^{-1}N^T\right]\right\}\eta(t)\Bigg\} \\ & \qquad + \mathbb{E}\left\{\sum_{i=1}^{2^m}\sum_{j=1}^{m}\delta_j^2 x^T(t)(B_1F_j\mathcal{K}_i)^T\mathscr{R}(B_1F_j\mathcal{K}_i)x(t)\right\}. \end{aligned}$$

From Lemma 2 and Schur complement, we can know that (15) is the sufficient condition to guarantee

$$\mathscr{L}V(x_t) + \mathbb{E}\{z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t)\} < 0. \tag{25}$$

When $\omega(t) = 0$, it means $\mathscr{L}V(x_t) < 0$; therefore, system (6) is mean asymptotically stable in the case of $\omega(t) = 0$. When $\omega(t) \neq 0$ is integrating both sides of (25), from 0 to t, yields

$$\mathbb{E}\left\{V(t) - V(0) + \int_0^t z^T(s)z(s)ds - \gamma^2\int_0^t \omega^T(t)\omega(t)\right\} < 0. \tag{26}$$

Letting $t \to \infty$ and under zero initial condition, we can show from (26) that

$$\mathbb{E}\left\{\int_0^\infty z^T(s)z(s)ds\right\} < \mathbb{E}\left\{\int_0^\infty \gamma^2\omega^T(s)\omega(s)ds\right\}. \tag{27}$$

This completes the proof. □

DOI: 10.1002/oca

Theorem 1 presents a sufficient condition which guarantees the mean-square asymptotically stability of the closed-loop system (6). Now, we will give an LMI-based optimization algorithm to obtain the largest contractively invariant ellipsoid $\mathscr{E}(P,1)$ for (6). By optimization method in [3], we can find an exact invariant set with least degree of conservativeness, which can be formulated as

$$\begin{aligned}&\max \alpha \\ &s.t. \begin{cases} \alpha\Omega \subset \mathscr{E}(P,1) \\ (14)--(15), \end{cases}\end{aligned} \tag{28}$$

where $\Omega = \mathscr{E}(\Pi, 1), \Pi \in \mathbb{R}^{n\times n}$.

Theorem 2

For given scalars $\tau_1, \tau_2, \gamma, \delta_j (j \in \mathcal{I})$, and matrix $\bar{\Xi}$. The closed-loop system (6), under all possible stochastic actuator failure and saturation is mean-square asymptotically stable, if there exist matrices $X > 0, \bar{Q}_1 > 0, \bar{Q}_2 > 0, \bar{R}_1 > 0, \bar{R}_2 > 0$ and scalars $\epsilon_i (i = 0, 1, \cdots, m)$ such that the following LMIs hold

$$\begin{bmatrix} \bar{u}_j & * \\ l_j^T & \bar{u}_j X \end{bmatrix} > 0 (j \in \mathcal{I}), \tag{29}$$

$$\begin{bmatrix} \bar{\Psi}_i^{11} + \bar{\Phi} + \bar{\Phi}^T & * & * & * & * \\ \bar{\Psi}_i^{21} & -2\epsilon_0 X + \epsilon_0^2 \bar{\mathscr{R}} & * & * & * \\ \bar{\Psi}_i^{31} & 0 & \bar{\Psi}^{33} & * & * \\ \bar{\Psi}_i^{41}(r) & 0 & 0 & -\bar{R}_2 & * \\ \bar{\Psi}^{51} & 0 & 0 & 0 & -I \end{bmatrix} < 0\,(i = 1, 2, \cdots, 2^m, r = 1, 2)\,, \tag{30}$$

where

$$\bar{\Psi}_i^{11} = \begin{bmatrix} \bar{\Gamma}_i & * & * & * & * \\ \bar{R}_1 & -\bar{Q}_1 - \bar{R}_1 & * & * & * \\ XA_d^T & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \\ B_2^T & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix},$$

$$\begin{aligned}\bar{\Gamma}_i = {} & AX + XA^T + B_1\bar{\Xi}W_iY + Y^TW_i^T\bar{\Xi}^TB_1^T + B_1\bar{\Xi}W_i^-L \\ & + L^TW_i^{-T}\bar{\Xi}^TB_1^T + \bar{Q}_1 + \bar{Q}_2 - \bar{R}_1,\end{aligned}$$

$$\bar{\Phi} = \begin{bmatrix} 0 & \bar{M} & \bar{N} - \bar{M} & -\bar{N} & 0 \end{bmatrix},$$

$$\bar{\Psi}_i^{21} = \begin{bmatrix} AX + B_1\bar{\Xi}(W_iY + W_i^-L) & 0 & A_dX & 0 & B_2X \end{bmatrix},$$

$$\bar{\Psi}_i^{31} = \begin{bmatrix} \delta_1 B_1 F_1(W_iY + W_i^-L) & 0 & 0 & 0 & 0 \\ \vdots & & & & \vdots \\ \delta_m B_1 F_m(W_iY + W_i^-L) & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{\Psi}_i^{33} = -diag\{\underbrace{-2\epsilon_1 X + \epsilon_1^2\bar{\mathscr{R}} \cdots - 2\epsilon_m X + \epsilon_m^2\bar{\mathscr{R}}}_{m}\}, \quad \bar{\mathscr{R}} = \tau_1^2\bar{R}_1 + (\tau_2 - \tau_1)\bar{R}_2,$$

$$\Psi^{51} = [CX0000], \bar{\Psi}_i^{41}(1) = \sqrt{\tau_2 - \tau_1}\bar{M}^T, \bar{\Psi}_{ij}^{41}(2) = \sqrt{\tau_2 - \tau_1}\bar{N}^T.$$

Moreover, the controller gain is given by $K = YX^{-1}$.

DOI: 10.1002/oca

Proof
From [3], the constraint (14) is equivalent to

$$\begin{bmatrix} \bar{u}_i & * \\ h_i^T & \bar{u}_i P \end{bmatrix} \geqslant 0 i \in \mathcal{I}, \tag{31}$$

where h_i is a i-th row of H.

Defining $X = P^{-1}$ and $l_i = h_i X$, by Schur complement, it can be further expressed as (29).

Besides, by Schur complement, Equation (15) is equivalent to

$$\begin{bmatrix} \hat{\Psi}_i^{11} + \Phi + \Phi^T & * & * & * & * \\ \Psi_i^{21} & -P\mathscr{R}^{-1}P & * & * & * \\ \Psi_i^{31} & 0 & \Psi^{33} & * & * \\ \Psi_i^{41}(r) & 0 & 0 & -R_2 & * \\ \Psi^{51} & 0 & 0 & 0 & -I \end{bmatrix} < 0 \left(\begin{array}{c} i = 1, 2, \cdots, 2^m \\ j \in \mathcal{I}; l = 1, 2 \end{array} \right), \tag{32}$$

where $\Psi^{33} = diag\{\underbrace{-P\mathscr{R}^{-1}P \cdots - P\mathscr{R}^{-1}P}_{m}\}, \Psi^{51} = [C \;\; 0 \;\; 0 \;\; 0 \;\; 0]$, and $\hat{\Psi}_i^{11}$ is the result of removing $C^T C$ in Γ_i in Ψ_i^{11}.

Owing to Equation (33) holds [18],

$$-P\mathscr{R}^{-1}P \leqslant -2\epsilon P + \epsilon^2 \mathscr{R}. \tag{33}$$

We have (32) hold if

$$\begin{bmatrix} \hat{\Psi}_i^{11} + \Phi + \Phi^T & * & * & * & * \\ \Psi_i^{21} & -2\epsilon_0 P + \epsilon_0^2 \mathscr{R} & * & * & * \\ \Psi_i^{31} & 0 & \tilde{\Psi}^{33} & * & * \\ \Psi_i^{41}(r) & 0 & 0 & -R_2 & * \\ \Psi^{51} & 0 & 0 & 0 & -I \end{bmatrix} < 0 \left(\begin{array}{c} i = 1, 2, \cdots, 2^m \\ j \in \mathcal{I}; l = 1, 2 \end{array} \right), \tag{34}$$

where $\tilde{\Psi}^{33} = diag\{\underbrace{-2\epsilon_1 P + \epsilon_1^2 \mathscr{R} \cdots - 2\epsilon_m P + \epsilon_m^2 \mathscr{R}}_{m}\}$.

Premultiply and post-multiply (34) by diag $\{X, X, X, X, I, X, \underbrace{X, \cdots, X}_{m}, X, I\}$ and define $\bar{R}_i = XR_iX, \bar{Q}_i = XQ_iX \;\; (i = 1, 2), \bar{M}_i = XM_iX, \bar{N}_i = XN_iX (i = 1, 2, 3, 4)$ and let $L = HX$, it follows that (30) holds. This completes the proof. □

Remark 4
To avoid the complications, we use the inequality (33) to convert nonlinear matrix inequalities (15) into LMIs; however, this step can be improved by adopting the cone complementary algorithm [19], which is popular in recent control designs.

This completes the proof.

To find an exact invariant set with least degree of conservativeness, which can be formulated as

$$\begin{array}{l} \max \;\; \alpha \\ s.t. \left\{ \begin{array}{l} \alpha \Omega \subset \mathscr{E}(P, 1) \\ (29) - -(30), \end{array} \right. \end{array} \tag{35}$$

DOI: 10.1002/oca

where $\Omega = \mathscr{E}(\Pi, 1)$, $\Pi \in \mathbb{R}^{n\times n}$. By Schur complement, it is equivalent to

$$\begin{aligned} & inf\ \beta \\ & s.t. \left\{ \begin{array}{l} \left[\begin{array}{cc} \beta\Pi & * \\ I & X \end{array} \right] \geqslant 0 \\ (29)--(30), \end{array} \right. \end{aligned} \tag{36}$$

where $\beta = 1/\alpha^2$.

Remark 5

In order to find the largest estimate for the domain of attraction, Zhang *et al.*[20] let $\|x(t)\| = \|\dot{x}(t)\|$, however, this condition is difficult to satisfy. The algorithm used in this paper can be easily converted to the forms of LMIs, and the estimated result is close to the true one; moreover, the obtained controller gain is much smaller, which will be illustrated in Section 4.

4. NUMERICAL EXAMPLE

Now, let us consider the two following illustrative examples to show the importance of our results. The first is to show the effectiveness of enlarging the domain of attraction, and the second one is provided to check the validness of the results dealing with the systems with both stochastic actuator failures and interval time-varying delay.

Example 1

Consider a linear state-delayed system (1) with the following matrices:

$$A = \left[\begin{array}{cc} 0.5 & -1 \\ 0.5 & -0.5 \end{array} \right], A_d = \left[\begin{array}{cc} 0.6 & 0.4 \\ 0 & -0.5 \end{array} \right], B_1 = \left[\begin{array}{c} 1 \\ 1 \end{array} \right], \bar{u}_i = 5.$$

Assume that there are no any actuator failure occurring; that is, $\bar{\Xi} = I_{m\times m}, \delta_i = 0\ (i \in \mathcal{I})$, and the time-varying delay satisfies $0 = \tau_1 < \tau(t) < \tau_2 = 0.35$. By using optimization algorithm (36), we obtain the following results:

$$P = \left[\begin{array}{cc} 0.0963 & -0.0165 \\ -0.0165 & 0.0200 \end{array} \right], K = \left[\begin{array}{cc} -1.9714 & 0.6422 \end{array} \right], H = \left[\begin{array}{cc} -1.4617 & 0.4696 \end{array} \right].$$

Figure 1 illustrates the largest invariant ellipsoid of the example by optimization problems (36). The inner dotted-dash ellipsoid is obtained by the method of [21], and the outer solid and dashed ellipsoid are the sets $\mathscr{E}(P, 1)$ and αX_R, respectively. It is clearly observed that the state of the examined system converges to the origin within the estimated domain of attraction despite the actuator saturation and interval time-varying delays; furthermore, it can be shown that our approach gives a lager estimation of the domain of attraction and a smaller state feedback gain. From Figure 1, we can also find that the ellipsoid set is contained inside the set of admissible saturations $\mathcal{L}(H)$.

If the time delay is a constant delay, that is, $\tau_1 = \tau_2 = \tau$, we can obtain the results listed in Table I by introducing the index δ_{max} as [20]. From Table I, we can find that our results are with less conservativeness because of using Lemma 2.

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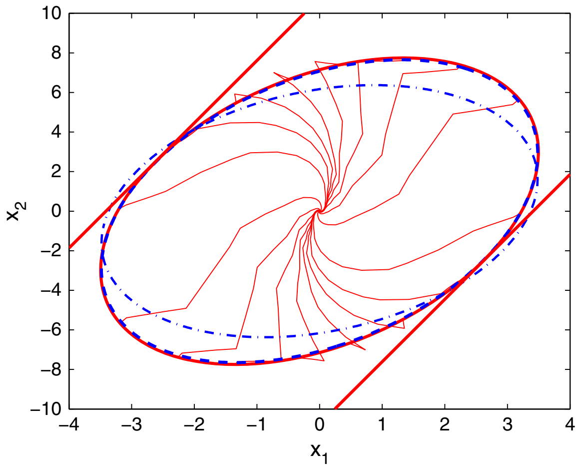

Figure 1. Estimates of the domain of attraction and state trajectories without actuator failure.

Table I. Computation results of Example 1.

Methods	$\tau = 0.35$	$\tau = 1.0$
Corllary 1 in [20]	6.0044	2.4571
Our results	6.1057	2.4623

Example 2
Consider the following linear time-varying delay systems (1) with

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}, A_d = \begin{bmatrix} -1 & 0.5 \\ -0.5 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

and the time-varying delay $\tau(t)$ satisfies $0.1 \leqslant \tau(t) \leqslant 1.2$, the saturation parameter is $\bar{u}_i = 2.5$, and the disturbance input $\omega(t) = e^{\frac{(t-6)^2}{0.125}}$, which is shown in Figure 2.

Stochastic failure of the actuator is considered in this example, and the fault model (4) with the parameters as $\bar{\Xi} = 0.72, \delta_1 = 0.2$. By using optimization algorithm (36) with $\gamma_{min} = 3.5$, the

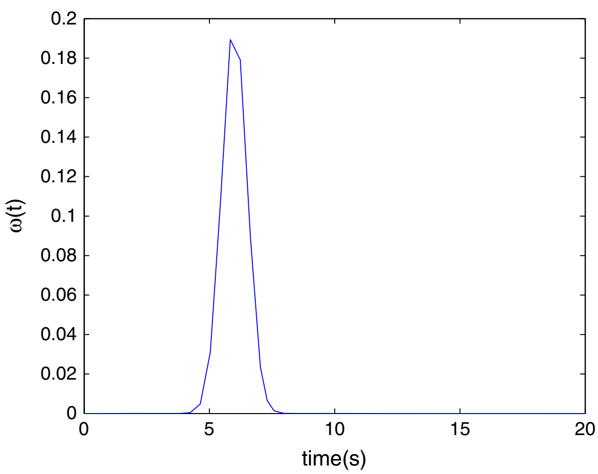

Figure 2. Disturbance input $\omega(t)$.

DOI: 10.1002/oca

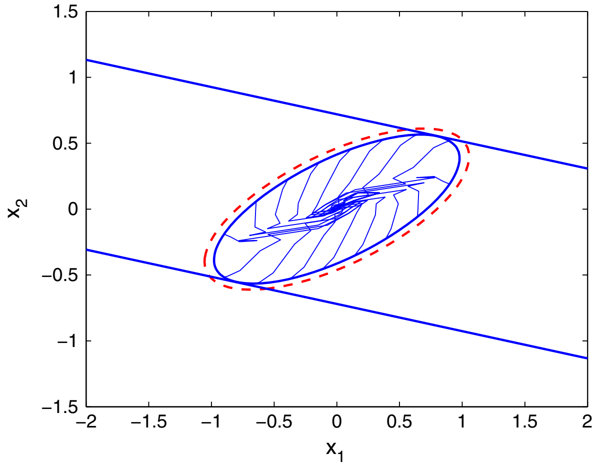

Figure 3. Estimates of the domain of attraction under actuator failure.

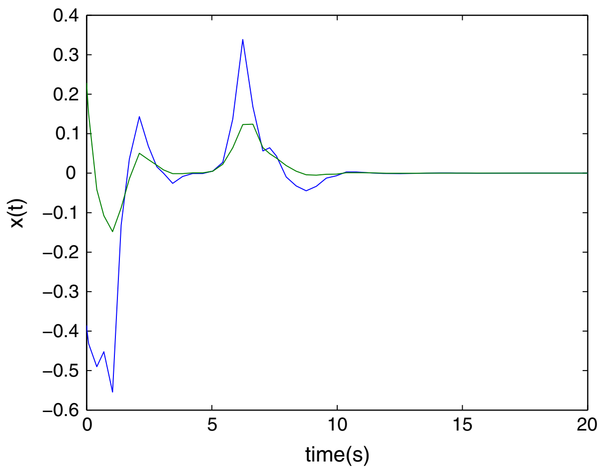

Figure 4. State trajectories using fault-tolerant controller.

following results can be obtained

$$P = \begin{bmatrix} 1.9464 & -2.2986 \\ -2.2986 & 5.8430 \end{bmatrix},$$

$$K = \begin{bmatrix} -1.0159 & -4.9367 \end{bmatrix},$$

$$H = \begin{bmatrix} -0.7164 & -3.4726 \end{bmatrix}.$$

Figure 3 depicts the resulting invariant ellipsoids for the system with normal (the outer) and stochastic actuator failure (the inner) conditions, respectively. It is shown that the ellipsoid becomes smaller by reasons of actuator failure effecting on the system. The trajectories of the closed-loop system using fault-tolerant controller is shown in Figure 4 with the initial conditions as $x(t) = [-0.3857\ 0.2285]^T \quad (t \in [-1.2\ 0])$.

DOI: 10.1002/oca

5. CONCLUSION

In this paper, the reliable control for a class of interval time-varying delay systems subject to stochastic actuator failure and saturation is investigated. A new stochastic fault model is proposed by introducing a random Ξ as a fault scale factor matrix. Delay distribution and fault distribution-dependent optimization approaches are proposed to enlarge the estimation of the domain of attraction by a set of LMIs. Numerical examples are used to show the effectiveness of our proposed method.

ACKNOWLEDGEMENT

This work was supported by the National Natural Science Foundation of China (Grant No. 61074025, 61074024, 60904013) and the Natural Science Foundation of the Jiangsu Higher Education Institutions of China (Grant No. 10KJB510007).

REFERENCES

1. Benzaouia A, Burgat C. Regulator problem for linear discrete-time systems with non-symmetrical constrained control. *International Journal of Control* 1988; **48**(6):2441–2451.
2. Tarbouriech S, Prieur C, da Silva Jr JMG. Stability analysis and stabilization of systems presenting nested saturations. *IEEE Transactions on Automatic Control* 2006; **51**(8):1364–1371.
3. Hu T, Lin Z, Chen B. An analysis and design method for linear systems subject to actuator saturation and disturbance* 1. *Automatica* 2002; **38**(2):351–359.
4. Peng C, Tian Y, Tadé M. State feedback controller design of networked control systems with interval time-varying delay and nonlinearity. *International Journal of Robust and Nonlinear Control* 2008; **18**(12):1285–1301.
5. Yue D, Han Q. Delayed feedback control of uncertain systems with time-varying input delay. *Automatica* 2005; **41**(2):233–240.
6. Tian E, Peng C. Delay-dependent stability analysis and synthesis of uncertain TS fuzzy systems with time-varying delay. *Fuzzy Sets and Systems* 2006; **157**(4):544–559.
7. Vidyasagar M, Viswanadham N. Reliable stabilization using a multi-controller configuration. *Automatica* 1985; **21**(5):599–602.
8. Zhang K, Jiang B, Shi P. A new approach to observer-based fault-tolerant controller design for Takagi–Sugeno fuzzy systems with state delay. *Circuits, Systems, and Signal Processing* 2009; **28**(5):679–697.
9. Zhang K, Jiang B, Cocquempot V. Adaptive observer-based fast fault estimation. *International Journal of Control Automation and Systems* 2008; **6**(3):320–326.
10. Wu H, Zhang H. Reliable mixed fuzzy static output feedback control for nonlinear systems with sensor faults. *Automatica* 2005; **41**(11):1925–1932.
11. Aiqing Z, Huajing F. Robust reliable guaranteed cost control for nonlinear singular stochastic systems with time delay*. *Journal of Systems Engineering and Electronics* 2008; **19**(4):791–798.
12. Yu L. An LMI approach to reliable guaranteed cost control of discrete-time systems with actuator failure. *Applied Mathematics and Computation* 2005; **162**(3):1325–1331.
13. Lien C, Yu K. Robust reliable control for uncertain time-delay systems with IQC performance. *Journal of Optimization Theory and Applications* 2008; **138**(2):235–251.
14. Zuo Z, Ho D, Wang Y. Brief paper: fault tolerant control for singular systems with actuator saturation and nonlinear perturbation. *Automatica (Journal of IFAC)* 2010; **46**(3):569–576.
15. Shi T, Su H, Chu J. On stability and stabilisation for uncertain stochastic systems with time-delay and actuator saturation. *International Journal of Systems Science* 2010; **41**(5):501–509.
16. Gu K, Kharitonov V, Chen J. *Stability of Time-Delay Systems*. Birkhauser: Boston, MA, 2003.
17. He Y, Wang Q, Lin C, Wu M. Augmented Lyapunov functional and delay-dependent stability criteria for neutral systems. *International Journal of Robust and Nonlinear Control* 2005; **15**(18):923–933.
18. Gu Z, Yue D, Wang D, Liu J. Stochastic faulty actuator-based reliable control for a class of interval time-varying delay systems with Markovian jumping parameters. *Optimal Control Applications and Methods* 2011; **32**(3):313–327.
19. El Ghaoui L, Oustry F, AitRami M. A cone complementarity linearization algorithm for static output-feedback and related problems. *IEEE Transactions on Automatic Control* 1997; **42**(8):1171–1176.
20. Zhang L, Boukas E, Haidar A. Delay-range-dependent control synthesis for time-delay systems with actuator saturation. *Automatica* 2008; **44**(10):2691–2695.
21. Zuo Z, Wang Y, Zhang G. Stability analysis and controller design for linear time delay systems with actuator saturation. *American Control Conference, 2007. ACC'07*, New York, NY, 9–13 July 2007; 5840–5844.

DOI: 10.1002/oca